Vectors and the Dot Product

1. Are the following better described by vectors or scalars?

   (a) The cost of a Super Bowl ticket.

   (b) The wind at a particular point outside.

   (c) The number of students at Harvard.

   (d) The velocity of a car.

   (e) The speed of a car.

2. Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?

Definition. The dot product $\vec{v} \cdot \vec{w}$ of two vectors $\vec{v}$ and $\vec{w}$ is defined as follows.

- If $\vec{v}$ and $\vec{w}$ are two-dimensional vectors, say $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2$.
- If $\vec{v}$ and $\vec{w}$ are three-dimensional vectors, say $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2 + v_3 w_3$.

It is not possible to dot a two-dimensional vector with a three-dimensional vector!

3. (a) What is $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$?

   (b) What is $\langle 1, 2, 3 \rangle \cdot \langle 4, -5, 6 \rangle$?

Here are some basic algebraic properties of the dot product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
3. $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\vec{w})$.

4. True or false: if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors of the same dimension, then $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.

5. What is the relationship between $\vec{v} \cdot \vec{v}$ and $|\vec{v}|$?

6. Find the angle between $\langle 1, 2, 1 \rangle$ and $\langle 1, -1, 1 \rangle$.

7. Find the vector projection of $\langle 0, 0, 1 \rangle$ onto $\langle 1, 2, 3 \rangle$.

8. True or false: If $\vec{v}$ and $\vec{w}$ are parallel, then $|\vec{v} - \vec{w}| = |\vec{v}| - |\vec{w}|$.

9. If $\vec{v}$ and $\vec{w}$ are vectors with the property that $|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$, which of the following must be true?

   (a) $\vec{v} = \vec{w}$.

   (b) $\vec{v} = \vec{0}$.

   (c) $\vec{v}$ is orthogonal to $\vec{w}$.

   (d) $\vec{v}$ is parallel to $\vec{w}$.